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PRECIPITOUSNESS IN FORCING EXTENSIONS

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ABSTRACT
It is shown that the precipitousness of ω_1 can persist in a variety of forcing extensions.

While the statement that there exists a cardinal carrying a precipitous ideal is equiconsistent with the statement that there exists a measurable cardinal, precipitous ideals can exist on nonmeasurable cardinals, for example, ω_1 can carry a precipitous ideal ([5]). Now if the GCH holds below a measurable then it holds at the measurable; similarly, if there is an $\aleph_2$-saturated ideal on ω_1 and $2^{\aleph_0} = \aleph_1$ then $2^{\aleph_1} = \aleph_2$. But the existence of a precipitous ideal on ω_1 is not known to have any such combinatorial consequences; a precise question in this vein (much less inclusive questions are also open, see below) is whether it is consistent that for every poset $\mathcal{P}$, $V^{\mathcal{P}} \models$ "every regular uncountable cardinal carries a precipitous ideal". In this paper we study the preservation in forcing extensions of the existence of a precipitous ideal on ω_1.

Some forcing extensions are known not to destroy the existence of a precipitous ideal on ω_1, but in the model where the measurable cardinal of an L_U has been Levy collapsed to ω_1, certain further extensions, which don't collapse ω_1, contain no precipitous ideals on ω_1. We study here what happens in the model V where a supercompact cardinal has been collapsed to ω_1, and prove for a certain class $\mathcal{A}$ of posets that $\mathcal{P} \in \mathcal{A}$ implies $V^{\mathcal{P}} \models \omega_1$ carries a precipitous ideal. $\mathcal{A}$ is a subclass of the class of proper posets. One of the members of $\mathcal{A}$ is the poset for making the proper forcing axiom true; thus the existence of a precipitous ideal on ω_1 is consistent with the proper forcing axiom. Is it consistent that the existence of a precipitous ideal on ω_1 is preserved under

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countably closed extensions? (Proper extensions? Extensions which don't collapse ω_1?)

After reviewing material on precipitous ideals and master conditions, the main result is proved. Then a limitation on those methods is given by another theorem, that over the model where a supercompact cardinal has been collapsed to ω_1, there is a countably closed extension in which the natural master condition ideals are not precipitous.

We recall some notation and definitions. Let κ be an uncountable cardinal and let $\mathcal{I} \subseteq \mathcal{P}(\kappa)$ be a κ-ideal, i.e., $\mathcal{I}$ is a κ-complete ideal, $\{\alpha\} \in \mathcal{I}$ for $\alpha < \kappa$, and $\kappa \notin \mathcal{I}$. Let $\mathcal{I}^+ = \mathcal{P}(\kappa) - \mathcal{I}$. Let $\Vdash_{\mathcal{I}}$ denote forcing with respect to the poset $(\mathcal{I}^+, \supseteq)$ and let $G_{\mathcal{I}}$ be the generic ultrafilter on $\mathcal{P}(\kappa) \cap V$ so obtained. In $V[G_{\mathcal{I}}]$, form the ultraproduct $V^\kappa / G_{\mathcal{I}}$, where V^κ is the class of functions $f : \kappa \to V$ with $f \in V$. Then $\mathcal{I}$ is *precipitous* if and only if $\Vdash_{\mathcal{I}}$ "$V^\kappa / G_{\mathcal{I}}$ is well founded". This notion of generic ultraproduct, and the question of its well foundedness, are due to Solovay ([15]), who proved that if $\mathcal{I}$ is κ^+-saturated then $\mathcal{I}$ is precipitous. Precipitousness was first studied for its own sake by Jech and Prikry (see [5]) who showed, by modifying deep methods of Kunen ([7]), that *if κ carries a precipitous ideal then κ is measurable in an inner model.*

Jech ([4]) showed that a game condition on $\mathcal{I}$ invented by Galvin is equivalent to precipitousness. Specifically, players I and II alternately move to create a sequence $Y_0 \supseteq Y_1 \supseteq \cdots \supseteq Y_n \supseteq \cdots$ $(n < \omega)$, with each $Y_n \in \mathcal{I}^+$; player I wins just in case $\bigcap Y_n = \emptyset$. Then *$\mathcal{I}$ is precipitous if and only if player I does not have a winning strategy.* Another game theoretic version, clearly equivalent to precipitousness (Hajnal and Shelah), is the following:

I	(X_0, f_0)		(X_1, f_1)		(X_2, f_2)	
II		Y_0		Y_1		$\cdots$

where $X_0 \supseteq Y_0 \supseteq X_1 \supseteq Y_1 \supseteq \cdots$, each $X_i, Y_i \in \mathcal{I}^+$, $f_i : X_i \to \mathrm{Ord}$, and $f_{i+1}(\alpha) < f_i(\alpha)$ for all $\alpha \in X_{i+1}$. Player I wins just in case the game continues for ω moves. Then *$\mathcal{I}$ is precipitous if and only if player II has a winning strategy.*

Say that κ is precipitous if there exists a precipitous κ-ideal. Mitchell ([5]) proved a converse to Jech and Prikry's result: *if a measurable cardinal κ is Levy collapsed to ω_1, then in the extension $\kappa = \omega_1$ is precipitous.* Thus, consider the following question. Let $\mathcal{L}^\kappa$ be the Levy collapse poset for making $\kappa = \omega_1$, where κ is measurable in V. Let $\mathcal{Q} \in V^{\mathcal{L}^\kappa}$ be a poset which doesn't collapse ω_1. Then is ω_1 precipitous in $V^{\mathcal{L}^\kappa * \mathcal{Q}}$? We have the following three facts.

(i) A theorem of Kakuda ([6]) is that *if $\mathcal{I}$ is a precipitous λ-ideal then in any λ cc forcing extension the ideal $\mathcal{I}$ generated by $\mathcal{I}$ is precipitous.* Thus $\mathcal{Q}$ may be any ccc poset, and the precipitousness of ω_1 is compatible with, say, MA & $\neg$CH.

(ii) Magidor ([5]) showed that $\mathcal{Q}$ can be a certain interation of length ω_2 which turns the precipitous ideal on ω_1 in $V^{\mathcal{L}^\kappa}$ into the nonstationary ideal, and keeps it precipitous. So *Con*(*ZFC* + *there is a measurable cardinal*) implies *Con*(*ZFC* + NS_{ω_1} *is precipitous*).

(iii) If $V = L_U$, U a normal ultrafilter on κ, then the question has a negative answer. Namely, in $(L_U)^{\mathcal{L}^\kappa}$ let $\mathcal{Q}_0 = 2^{<\kappa} \cap L_U$ ordered by function extension, and let $\mathcal{Q}_1$ be the usual poset for mapping ω_1 onto ω_2 with countable conditions. Then in $(L_U)^{\mathcal{L}^\kappa * \mathcal{Q}_0}$ and $(L_U)^{\mathcal{L}^\kappa * \mathcal{Q}_1}$, ω_1 is not precipitous. These facts are consequences of Kunen's work ([7]) on L_U. Namely, suppose $\mathcal{I}$ is a precipitous ideal on ω_1 in $L_U[G, H]$, where G is L_u-$\mathcal{L}^\kappa$-generic and H is either $L_U[G]$-$\mathcal{Q}_0$-generic or $L_U[G]$-$\mathcal{Q}_1$-generic. Let $G_{\mathcal{I}}$ be a generic ultrafilter over $\mathcal{I}$, with

$$i : L_U[G, H] \to (L_U[G, H])^\kappa / G_{\mathcal{I}} = M$$

the canonical embedding. Then $M = L_{iU}[iG, iH]$, where L_{iU} is the unique "$i(\kappa)$-model" ([7]), i.e. the unique transitive model of ZFC containing all the ordinals satisfying "there is a normal ultrafilter D on $i(\kappa)$ with $V = L_D$".

In the case where H is $\mathcal{Q}_0$-generic, $M \models$ "$iH \notin L_{iU}$ but $iH \cap \alpha \in L_{iU}$ for all $\alpha < i(\kappa)$". Thus $M \models H \in L_{iU}$, since $iH \cap \kappa = H$. Whence $H \in L_{iU}$ by absoluteness of L_{iU}. But $L_U \subsetneq L_{iU}$ ([7]), so $H \in L_U$, a contradiction.

If H is $\mathcal{Q}_1$-generic, then, since $\mathcal{P}(\kappa) \cap L_U[G, H] \subseteq M$, and since $(\kappa^+)^{L_U} \sim \kappa$ in $L_U[G, H]$, we have $\mathcal{P}((\kappa^+)^{L_U}) \cap L_U \subseteq M$. Thus $U \in M$, so $M \models$ "there is a κ-model". But since all the λ-models are elementarily equivalent ([7]) the least λ-model would have a generic extension in which for some $\lambda' < \lambda$, there is a λ'-model, a contradiction.

Similar arguments produce a counterexample $\mathcal{Q}$ to the above question, when κ is a measurable cardinal in one of the Φ-minimal models of [12] or [13]. We consider the question of the precipitousness of ω_1 in certain $V^{\mathcal{L}^\kappa * \mathcal{Q}}$, where κ is supercompact.

Let $j : V \to M$ be an elementary embedding, M a transitive class, and let $\mathcal{P}$ be a poset. Suppose $G_{j\mathcal{P}}$ is V-$j\mathcal{P}$-generic, whence M-$j\mathcal{P}$-generic. Living in $V[G_{j\mathcal{P}}]$, we desire that $j^{-1}{}''(G_{j\mathcal{P}})$ be V-$\mathcal{P}$-generic and that j be lifted to an elementary embedding $\tilde{j} : V[j^{-1}{}''(G_{j\mathcal{P}})] \to M[G_{j\mathcal{P}}]$.

DEFINITION (See Kunen and Paris ([9]), Silver ([11])). $\mathcal{P}$ has the *master condition property* for j if the following equivalent conditions hold:

(i) $\forall p \in \mathcal{P}\ \exists r \in j\mathcal{P}\ r \geqq jp$ and $r \Vdash_{j\mathcal{P}} j^{-1}{}''(G_{j\mathcal{P}})$ is V-$\mathcal{P}$-generic,

(ii) $\Vdash_{\mathcal{P}} j''(G_{\mathcal{P}})$ determines a V-generic set with respect to the subalgebra of $j\mathcal{P}$ generated by $j''\mathcal{P}$.

The r of (i) is a master condition. If M is sufficiently closed then the "$\Vdash_{j\mathcal{P}}$" of (i) may be taken as forcing over M and the "V-generic" of (ii) may be replaced by "M-generic". To see that (i) implies (ii), take a $p \in \mathcal{P}$ forcing the negation of the statement to be forced in (ii), and apply (i) to get a contradiction. For (ii) implies (i), take a $p \in \mathcal{P}$ contradicting (i) and extend $j''(G_{\mathcal{P}})$ to a generic set on $j\mathcal{P}$.

Lemma 1 ([9], [11]). *If $G_{j\mathcal{P}}$ is $j\mathcal{P}$-generic, and $G_{\mathcal{P}} = j^{-1}{}''(G_{j\mathcal{P}})$ is $\mathcal{P}$-generic, then in $V[G_{j\mathcal{P}}]$, j may be extended to an elementary embedding $\tilde{j} : V[G_{\mathcal{P}}] \to M[G_{j\mathcal{P}}]$.*

Proof. If τ is a $\mathcal{P}$-term, let $\tilde{j}(\mathrm{den}_{G_{\mathcal{P}}} \tau) = \mathrm{den}_{G_{j\mathcal{P}}}(j\tau)$. This is well defined since if $p \in G_{\mathcal{P}}$ and $p \Vdash_{\mathcal{P}} \tau = \sigma$ then $jp \in G_{j\mathcal{P}}$ and $jp \Vdash_{j\mathcal{P}} j\tau = j\sigma$; similarly, the embedding is elementary.

Remarks. If $\mathcal{P}$ has the κ cc, where κ is the critical point of j, then trivially $\mathcal{P}$ has the master condition property for j (take $r = jp$). Kunen and Paris ([9]) first considered extending elementary embeddings in cases where $\mathcal{P}$ doesn't have the κ cc. Silver ([11]) was the first to utilize a nontrivial master condition. Master conditions in situations where $\mathcal{P}$ collapses κ to a successor cardinal were first used by Kunen ([8]).

Lemma 2 ([9], [11], [5]). *Let κ be the critical point of j and suppose $\mathcal{P}$ has the master condition property for j. In $V[G_{\mathcal{P}}]$ define, for $X \subseteq \kappa$,*

$$X \in \mathcal{I} \Leftrightarrow \exists p \in G_{\mathcal{P}}\ jp \Vdash_{j\mathcal{P}} (j^{-1}{}''(G_{j\mathcal{P}}) \text{ is } V\text{-}\mathcal{P}\text{-generic} \Rightarrow \kappa \notin \tilde{j}X).$$

Then $\mathcal{I}$ is a normal κ-ideal.

Proof. That $\kappa \notin \mathcal{I}$ follows from the master condition property. That $\mathcal{I}$ is an ideal is quickly verified. If $\mathcal{I}$ weren't normal, then for some q, $\dot{X}$ and $\dot{f}$,

$$q \Vdash_{\mathcal{P}} \dot{X} \in \mathcal{I}^+, \qquad \dot{f}(\beta) < \beta \quad (\text{all } \beta \in \dot{X}), \qquad \dot{f}^{-1}\{\alpha\} \in \mathcal{I} \quad (\text{all } \alpha < \kappa).$$

Choose an $\alpha < \kappa$ and $s \in j\mathcal{P}$ so that $s \geqq jq$ and

$$s \Vdash_{j\mathcal{P}} j^{-1}{}''(G_{j\mathcal{P}}) \text{ is } V\text{-}\mathcal{P}\text{-generic}, \quad \kappa \in \tilde{j}\dot{X}, \quad \text{and} \quad j\dot{f}(\kappa) = \alpha.$$

Take a generic $G_{j\mathcal{P}}$ containing s. Since $q \Vdash_{\mathcal{P}} \{\beta \in \dot{X} : \dot{f}(\beta) = \alpha\} \in \mathcal{I}$, a witness $p \geqq q$ for this, as in the definition of $\mathcal{I}$, may be found with $p \in j^{-1}{}''(G_{j\mathcal{P}})$. But this contradicts that $s \in G_{j\mathcal{P}}$.

Definition (Shelah [14], see also [1], [3]). A poset $\mathcal{Q}$ is *proper* if and only if the following equivalent conditions hold:

(i) $\forall\lambda$ $\forall$ stationary $\mathcal{S} \subseteq [\lambda]^{\aleph_0}$ $\mathcal{S}$ remains stationary in $V[G_{\mathcal{Q}}]$,
(ii) $\forall\delta > 2^{\mathrm{Card}\,\mathcal{Q}}$ $\forall$ countable $N \prec (H_\delta, \in, <)$ ($<$ a well ordering of H_δ) with $\mathcal{Q} \in N$ $\forall q \in \mathcal{Q} \cap N$ $\exists r \geqq q$ $\forall A \in N$ (A a maximal antichain of $\mathcal{Q} \Rightarrow r \Vdash G_{\mathcal{Q}} \cap A \in N$).

LEMMA 3. *Let $j : V \to M$ have critical point κ, and assume $\Vdash_{\mathcal{L}^\kappa} \mathcal{Q}$ is proper. For $\lambda = 2^{\mathrm{Card}\,\mathcal{L}^\kappa * \mathcal{Q}}$, suppose $[M]^\lambda \subseteq M$ and $j(\kappa) > \lambda$. Then $\mathcal{L}^\kappa * \mathcal{Q}$ has the master condition property for j.*

PROOF. Given $(p, \dot{q}) \in \mathcal{L}^\kappa * \mathcal{Q}$ we are to find a generic $G_{j(\mathcal{L}^\kappa * \mathcal{Q})}$ containing (p, jq) such that $j^{-1}{}''(G_{j(\mathcal{L}^\kappa * \mathcal{Q})})$ is $\mathcal{L}^\kappa * \mathcal{Q}$-generic. First pick a generic $G_{j(\mathcal{L}^\kappa)}$ containing p. We have the elementary embedding $\bar{j} : V[G_{\mathcal{Q}^\kappa}] \to M[G_{j\mathcal{Q}^\kappa}]$, where $G_{\mathcal{L}^\kappa} = G_{j\mathcal{L}^\kappa} \restriction \mathcal{L}^\kappa$. In $V[G_{\mathcal{L}^\kappa}]$, pick a well ordering $<$ of H_λ. Then by closure of M, $N = j''(H_\lambda, \in, <) \in M[G_{j\mathcal{L}^\kappa}]$; furthermore, $N \prec j(H_\lambda, \in, <)$. We have that $\bar{j}\mathcal{Q}$ is proper in $M[G_{j\mathcal{L}^\kappa}]$, $j\mathcal{Q}$, $jq \in N$ and, since $G_{j\mathcal{L}^\kappa}$ is the Levy collapse of $j(\kappa) > \lambda$ to ω_1, N is countable in $M[G_{j\mathcal{L}^\kappa}]$. Thus we may pick $r \geqq jq$ as guaranteed by properness of $j\mathcal{Q}$. Then for every maximal antichain A of $\mathcal{Q}$ in $V[G_{\mathcal{L}^\kappa}]$, $r \Vdash_{j\mathcal{Q}} G_{j\mathcal{Q}} \cap jA \in N$, whence $r \Vdash_{j\mathcal{Q}} j^{-1}{}''(G_{j\mathcal{Q}})$ is $V[G_{\mathcal{L}^\kappa}]$-$\mathcal{Q}$-generic, completing the proof.

For a poset $\mathcal{P}$, Mitchell's method, if it applies, for getting a precipitous ideal on κ in $V[G_{\mathcal{P}}]$, is to extend the embedding

$$i : V \to V^\kappa / U = M$$

given by a normal ultrafilter U on κ in V, to

$$\bar{i} : V[G_{\mathcal{P}}] \to M[G_{i\mathcal{P}}],$$

obtaining a κ-ideal $\mathcal{I}$ in $V[G_{\mathcal{P}}]$, and then to argue that

$$(V[G_{\mathcal{P}}])^\kappa / G_{\mathcal{I}} \cong M[G_{i\mathcal{P}}]$$

whence $(V[G_{\mathcal{P}}])^\kappa / G_{\mathcal{I}}$ is well founded. In the arguments of Magidor ([5], the result that NS_{ω_1} can be precipitous) and Baumgartner and Taylor ([2], starting with κ being precipitous rather than measurable) an isomorphism of this sort proves precipitousness.

However, when $\mathcal{P} = \mathcal{L}^\kappa * \mathcal{Q}$, $\mathcal{Q}$ proper in $V[G_{\mathcal{L}^\kappa}]$, $\mathcal{Q}$'s largeness may force us to use a supercompact embedding j and Lemmas 1 and 3 to get a $\bar{j} : V[G_{\mathcal{L}^\kappa * \mathcal{Q}}] \to M[G_{j(\mathcal{L}^\kappa * \mathcal{Q})}]$. What may then be established by Mitchell's method is that if κ is supercompact then for sufficiently large λ there is a precipitous ideal on $[\lambda]^{\aleph_0}$ in $V[G_{\mathcal{L}^\kappa * \mathcal{Q}}]$. But for $\lambda = \kappa$, the isomorphism, which would guarantee the precipitousness of the κ-ideal $\mathcal{I}$ defined as in Lemma 2, does not hold. In this paper we prove that sometimes $\mathcal{I}$ is precipitous anyway.

An improved statement of Lemma 3 is

LEMMA 4. *Let $j : V \to M$ have critical point κ, and assume $\Vdash_{\mathcal{L}^\kappa} \mathcal{Q}_0$ is proper, $\Vdash_{\mathcal{L}^\kappa * \mathcal{Q}_0} \mathcal{Q}_1$ is proper. For $\lambda = 2^{\mathrm{Card}(\mathcal{L}^\kappa * \mathcal{Q}_0 * \mathcal{Q}_1)}$, suppose $M^\lambda \subseteq M$ and $j(\kappa) > \lambda$. Let $G_{j(\mathcal{L}^\kappa * \mathcal{Q}_0)}$ be $j(\mathcal{L}^\kappa * \mathcal{Q}_0)$-generic with $j^{-1}{}''(G_{j(\mathcal{L}^\kappa * \mathcal{Q}_0)}) = G_{\mathcal{L}^\kappa * \mathcal{Q}_0}$ being $(\mathcal{L}^\kappa * \mathcal{Q}_0)$-generic (Lemma 3). Let $\dot{q}_1 \in \mathcal{Q}_1$. Then $G_{j(\mathcal{L}^\kappa * \mathcal{Q}_0)} \cup \{j\dot{q}_1\}$ may be extended to a $j(\mathcal{L}^\kappa * \mathcal{Q}_0 * \mathcal{Q}_1)$-generic $G_{j(\mathcal{L}^\kappa * \mathcal{Q}_0 * \mathcal{Q}_1)}$ so that $j^{-1}{}''(G_{j(\mathcal{L}^\kappa * \mathcal{Q}_0 * \mathcal{Q}_1)})$ is $(\mathcal{L}^\kappa * \mathcal{Q}_0 * \mathcal{Q}_1)$-generic.*

PROOF. As in Lemma 3. Let

$$\bar{j} : V[G_{\mathcal{L}^\kappa * \mathcal{Q}_0}] \to M[G_{j(\mathcal{L}^\kappa * \mathcal{Q}_0)}]$$

be the canonical elementary embedding. In $V[G_{\mathcal{L}^\kappa * \mathcal{Q}_0}]$, pick a well ordering $<$ of H_λ. Then $N = j''(H_\lambda, \in, <)$ is, in $M[G_{j(\mathcal{L}^\kappa * \mathcal{Q}_0)}]$, a countable elementary substructure of $j(H_\lambda, \in, <)$ containing $\bar{j}(\mathcal{Q}_1)$. So a master condition in $\bar{j}\mathcal{Q}_1$ (extending jq_1) exists as in Lemma 3.

Say that a poset $\mathcal{R}$ is *strongly generated by $\aleph_1$ antichains* if there are antichains A_α of $\mathcal{R}$ ($\alpha < \omega_1$) with $\bigcup_{\alpha<\omega_1} A_\alpha$ cofinal in $\mathcal{R}$. Let $\mathcal{A}$ be the class of posets $\mathcal{Q}$ such that

(i) $\mathcal{Q}$ is proper.

(ii) If B_α ($\alpha < \omega_1$) are antichains of $\mathcal{Q}$ then $\mathcal{Q}$ may be written as $\mathcal{R} * \mathcal{S}$, with $\mathcal{S}/G_{\mathcal{R}}$ proper, with each $B_\alpha \subseteq \mathcal{R}$, and with $\mathcal{R}$ strongly generated by $\aleph_1$ antichains.

$\mathcal{A}$ is the collection of proper posets which will be shown below to preserve the precipitousness of ω_1 over $V[G_{\mathcal{L}^\kappa}]$, κ supercompact. However, some improper posets are also known to preserve precipitousness:

(a) Magidor's poset $\mathcal{Q} \in V[G_{\mathcal{L}^\kappa}]$, κ measurable, such that NS_{ω_1} is precipitous in $V[G_{\mathcal{L}^\kappa * \mathcal{Q}}]$, is not proper.

(b) If κ is indestructibly supercompact ([10]), then in $V[G_{\mathcal{L}^\kappa}]$, while $\mathcal{Q} = 2^{<\kappa} \cap V$ is not proper, ω_1 remains precipitous upon forcing with $\mathcal{Q}$ (or in fact any $\mathcal{Q}' \in V$, $\mathcal{Q}'$ κ-directed closed in V).

$\mathcal{A}$ contains the ccc posets and the poset of countable conditions for blowing up 2^{ω_1} to a cardinal γ (these only require κ measurable to get precipitousness), proper posets of size $\aleph_1$, the poset for mapping ω_1 onto γ with countable conditions, and length ω_2, countable support, $\aleph_2$-cc, proper iterations such that each stage before ω_2 has a dense subset of size $\leqq \aleph_1$. We prove that another type of poset is in $\mathcal{A}$, by giving an example.

Shelah ([14]) proved that a countable support iteration of proper forcings is proper, thus obtaining versions of MA for proper forcings. Baumgartner ([1]) iterated proper forcings in a certain way δ times for a supercompact δ, concluding the consistency, from Shelah's methods, of

PFA: If $\mathcal{R}$ is proper and D_α is cofinal in $\mathcal{R}$ $(\alpha < \omega_1)$ then there is an $\mathcal{R}$-$\{D_\alpha : \alpha < \omega_1\}$-generic set.

Proper iterations of long length will be in $\mathcal{A}$ if enough cardinals are collapsed along the way. Let us check

LEMMA 5. *The standard posets $\mathcal{Q}$ for making PFA true are in $\mathcal{A}$.*

PROOF. $\mathcal{Q}$ is a certain length δ, δ cc iteration with countable supports (δ supercompact) of proper posets. The $\mathcal{Q}$'s ([1], [14]) which have been used for PFA satisfy that for cofinally many inaccessible $\alpha < \delta$, the poset $\mathcal{Q}_\alpha$ giving the first α steps of the iteration has α-cc and cardinality α, and $\mathcal{Q}_{\alpha+1} = \mathcal{Q}_\alpha * \mathcal{L}_\alpha$, where $\mathcal{L}_\alpha$ is the Levy poset for mapping ω_1 onto α. Suppose that B_β $(\beta < \omega_1)$ are antichains of $\mathcal{Q}$. Pick $\alpha < \delta$ as above with each $B_\beta \subseteq \mathcal{Q}_\alpha$. Let $\mathcal{R} = \mathcal{Q}_{\alpha+1}$. Then $\mathcal{Q}/G_{\mathcal{R}}$ is proper ([14]). The conditions on α above imply that $\operatorname{Card} \mathcal{R} = \alpha$. Enumerate $\mathcal{R}$ as $\{r_\sigma : \sigma < \alpha\}$. Let $g : \omega_1 \rightarrow \alpha$ be the generic map given by $\mathcal{L}_\alpha$. For each $\beta < \omega_1$ let A_β be a maximal antichain of $\mathcal{R}$ such that for each $\sigma < \alpha$, if r_σ is compatible with $[\![\dot{g}(\beta) = \sigma]\!]$ then some extension of r_σ, $[\![\dot{g}(\beta) = \sigma]\!]$ is in A_β. Then $\bigcup_{\beta<\omega_1} A_\beta$ is cofinal in $\mathcal{R}$, since for each $\sigma < \alpha$, r_σ is compatible with $[\![\dot{g}(\beta) = \sigma]\!]$ for some β. Thus $\mathcal{R}$ is strongly generated by $\aleph_1$ antichains.

LEMMA 6. *If $\Vdash_{\mathcal{L}^\kappa} \mathcal{R}$ is strongly generated by $\aleph_1$ antichains then $\mathcal{L}^\kappa * \mathcal{R}$ is strongly generated by κ antichains.*

PROOF. If $\Vdash \bigcup_{\alpha<\kappa} \dot{A}_\alpha$ is cofinal in $\dot{\mathcal{R}}$, with $\dot{A}_\alpha = \{\dot{a}_{\alpha\gamma} : \gamma < \sigma_\alpha\}$ an enumeration without repetitions, then $\bigcup_{\alpha<\kappa, p \in \mathcal{L}^\kappa} \{(p, \dot{a}_{\alpha\gamma}) : \gamma < \sigma_\alpha\}$ is cofinal in $\mathcal{L}^\kappa * \dot{\mathcal{R}}$.

THEOREM. *Suppose κ is supercompact and $\Vdash_{\mathcal{L}^\kappa} \dot{\mathcal{Q}} \in \mathcal{A}$. Then in $V[G_{\mathcal{L}^\kappa * \mathcal{Q}}]$, ω_1 is precipitous.*

PROOF. Let $G = G_{\mathcal{L}^\kappa * \mathcal{Q}}$; we show the ideal $\mathcal{I}$ obtained from G and a sufficiently closed supercompact embedding j (Lemmas 1–3) is precipitous. Using the second game theoretic version of precipitousness in the introduction, player II must respond to an (X, f) $(X \in \mathcal{I}^+$ and $f : X \rightarrow \mathrm{Ord})$ with a $Y \subseteq X$, $Y \in \mathcal{I}^+$.

Player II picks a term, $(\dot{X}, \dot{f})$ for (X, f) and a decomposition $\mathcal{Q} = \mathcal{R} * \mathcal{S}$ as in property $\mathcal{A}$ for the maximal antichains needed to decide $\dot{X}$ and $\dot{f}$. Thus $(X, f) \in V[G_{\mathcal{L}^\kappa * \mathcal{R}}]$, $\Vdash_{\mathcal{L}^\kappa * \mathcal{R}} \mathcal{S}$ is proper, and $\Vdash_{\mathcal{L}^\kappa} \mathcal{R}$ is strongly generated by $\aleph_1$

antichains, whence by Lemma 6, $\mathcal{L}^\kappa * \mathcal{R}$ is strongly generated by maximal antichains $\{A_\alpha : \alpha < \kappa\}$.

If $\Vdash_{\mathcal{C}} \dot{h} : \dot{X} \to \mathrm{Ord}$ and $\mathcal{B}$ is a complete subalgebra of $\mathcal{C}$, and $G_{\mathcal{B}}$ is $\mathcal{B}$-generic then, assuming $[\![\alpha \in \dot{X}]\!]_{\mathcal{C}/G_{\mathcal{B}}} > 0$, let

$$\mu(\dot{h}, \alpha, G_{\mathcal{B}}) = \text{the least ordinal } \theta \text{ such that for some } c \in \mathcal{C}/G_{\mathcal{B}},\ c \Vdash \dot{h}(\alpha) = \theta.$$

For $\alpha < \kappa$ let $\mathcal{B}_\alpha$ be the complete subalgebra of $\mathcal{L}^\kappa * \mathcal{R}$ generated by $\bigcup_{\beta<\alpha} A_\beta$, and let $G_{\mathcal{B}_\alpha} = G \cap \mathcal{B}_\alpha$. Then player II plays

$$Y = \{\alpha \in X : f(\alpha) = \mu(\dot{f}, \alpha, G_{\mathcal{B}_\alpha})\}.$$

We show that $Y \in \mathcal{I}^+$ and that player II's strategy guarantees that I will have no legal move after some finite number of plays.

In M, let $\mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{R})}$ and $\mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{Q})}$ be the complete subalgebras of $j(\mathcal{L}^\kappa * \mathcal{Q})$ generated by $j''(\mathcal{L}^\kappa * \mathcal{R})$ and $j''(\mathcal{L}^\kappa * \mathcal{Q})$, respectively. If H is $j(\mathcal{L}^\kappa * \mathcal{Q})$-generic, define in $M[H]$

$$\theta_0 = \mu(j\dot{f}, \kappa, H \restriction \mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{Q})}),$$

$$\theta_1 = \mu(j\dot{f}, \kappa, H \restriction \mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{R})}),$$

$$\theta_2 = \mu(j\dot{f}, \kappa, H \restriction \mathcal{B}_\kappa),$$

where

$$\mathcal{B}_\kappa = (j\langle \mathcal{B}_\alpha : \alpha < \kappa\rangle)(\kappa).$$

Note that since $X \in \mathcal{I}^+$, θ_0, θ_1 and θ_2 exist whenever $j^{-1''}(H)$ is $(\mathcal{L}^\kappa * \mathcal{Q})$-generic, and that they are unchanged by replacing the term $\dot{f}$ by an $\dot{f}'$ with the same denotation in $V[j^{-1''}(H)]$.

Lemma 7. *Let H be $j(\mathcal{L}^\kappa * \mathcal{Q})$-generic and assume $j^{-1''}(H)$ is $\mathcal{L}^\kappa * \mathcal{Q}$-generic. Then in $M[H]$, $\theta_0 = \theta_1 = \theta_2$.*

Proof. $\theta_0 = \theta_1$. Assuming that $j^{-1''}(H)$ is $(\mathcal{L}^\kappa * \mathcal{Q})$-generic, the sets $H \restriction \mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{Q})}$ and $H \restriction \mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{R})}$ (and hence the values of θ_0, θ_1, respectively) are generated by their j inverse images, which are $(\mathcal{L}^\kappa * \mathcal{Q})$- and $(\mathcal{L}^\kappa * \mathcal{R})$-generic, respectively. So we may assume for a contradiction that there is a $(p, \dot{s}) \in (\mathcal{L}^\kappa * \mathcal{R}) * \mathcal{S}$ so that

$$j(p, \dot{s}) \Vdash_{j(\mathcal{L}^\kappa * \mathcal{Q})} \text{"if } j^{-1''}(H) \text{ is } (\mathcal{L}^\kappa * \mathcal{Q})\text{-generic, then } \theta_0 \neq \theta_1\text{"}.$$

Since $j\dot{f}(\kappa)$ is determined by forcing over $j(\mathcal{L}^\kappa * \mathcal{R})$, pick a generic $G_{(\mathcal{K}^\kappa * \mathcal{R})}$ containing p and extend $j''(G_{\mathcal{L}^\kappa * \mathcal{R}})$ to a generic $H_{j(\mathcal{L}^\kappa * \mathcal{R})}$ containing an h forcing that $j\dot{f}(\kappa) = \theta_1$. Now using Lemma 4, extend $H_{j(\mathcal{L}^\kappa * \mathcal{R})}$ to a $j(\mathcal{L}^\kappa * \mathcal{Q})$-generic H containing $j\dot{s}$ so that $j^{-1}{}''(H)$ is $\mathcal{L}^\kappa * \mathcal{Q}$-generic. But now $M[H]$ satisfies $\theta_0 = \theta_1$.

$\theta_1 = \theta_2$. Suppose $j^{-1}{}''(H_{j(\mathcal{L}^\kappa * \mathcal{R})}) = G_{\mathcal{L}^\kappa * \mathcal{R}}$ is $(\mathcal{L}^\kappa * \mathcal{R})$-generic. We claim that the set $\bigcup_{\alpha<\kappa} j''(G_{\mathcal{L}^\kappa * \mathcal{R}} \cap A_\alpha)$ is a subset of $\mathcal{B}_\kappa$ and $\mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{R})}$, and generates $H \restriction \mathcal{B}_\kappa$ and $H \restriction \mathcal{B}_{j''(\mathcal{L}^\kappa * \mathcal{R})}$. Since $\mathcal{B}_\kappa$ is generated by the antichains jA_α $(\alpha < \kappa)$, $\bigcup_{\alpha<\kappa} j''(G_{\mathcal{L}^\kappa * \mathcal{R}} \cap A_\alpha)$ generates $H \restriction \mathcal{B}_\kappa$. As $\bigcup_{\alpha<\kappa} A_\alpha$ is cofinal in $\mathcal{L}^\kappa * \mathcal{R}$, it follows that $\bigcup_{\alpha<\kappa} (G_{\mathcal{L}^\kappa * \mathcal{R}} \cap A_\alpha)$ is cofinal in $G_{\mathcal{L}^\kappa * \mathcal{R}}$. Hence $\bigcup_{\alpha<\kappa} j''(G_{\mathcal{L}^\kappa * \mathcal{R}} \cap A_\alpha)$ is cofinal in $j''(G_{\mathcal{L}^\kappa * \mathcal{R}})$, whence generates, since $j''(G_{\mathcal{L}^\kappa * \mathcal{R}})$ generates, $H \restriction \mathcal{B}_\kappa$. Note that we needed $\bigcup_{\alpha<\kappa} (G_{\mathcal{L}^\kappa * \mathcal{R}} \cap A_\alpha)$ to be cofinal in, rather than just generate, $G_{\mathcal{L}^\kappa * \mathcal{R}}$, to reach the conclusion that $H \restriction \mathcal{B}_\kappa$ is generated by applying j. The claim yields $\theta_1 = \theta_2$.

Given X, f, Y as above, let θ_f be the $\theta_0 = \theta_1 = \theta_2$ of Lemma 7.

LEMMA 8. *Let G be $(\mathcal{L}^\kappa * \mathcal{Q})$-generic, $Y \in V[G]$ defined from (X, f) as above. Then $Y \in \mathcal{I}^+$, and if $j''(G) \subseteq H$, H generic over $j(\mathcal{L}^\kappa * \mathcal{Q})$, then*

$$M[H] \models (\kappa \in \tilde{j}Y \Leftrightarrow \tilde{j}f(\kappa) = \theta_f).$$

PROOF. In $M[j''G]$ there is by definition a condition in $j(\mathcal{L}^\kappa * G)/j''(G)$ which forces $(\tilde{j}f)(\kappa) = \theta_0$, hence that $(\tilde{j}f)(\kappa) = \theta_2$, which is the definition of $\kappa \in \tilde{j}Y$.

To finish the theorem, suppose in $V[G_{\mathcal{L}^\kappa * \mathcal{Q}}]$ the game

I	(X_0, f_0)		(X_1, f_1)		
II		Y_0		Y_1	$\cdots$

continues for ω moves, with II playing his strategy. Live in $V[H]$, some H generic over $j(\mathcal{L}^\kappa * \mathcal{Q})$ with $j^{-1}{}''H = G_{\mathcal{L}^\kappa * \mathcal{Q}}$, where the embedding $\tilde{j} : V[G_{\mathcal{L}^\kappa * \mathcal{Q}}] \to M[H]$ is defined. Since each $X_n \in \mathcal{I}^+$, θ_{f_n} exists for each n. We claim that $\theta_{f_n} > \theta_{f_{n+1}}$. Namely, pick a condition s in $j(\mathcal{L}^\kappa * \mathcal{Q})/j''(G_{\mathcal{L}^\kappa * \mathcal{Q}})$ forcing that $\kappa \in \tilde{j}Y_{n+1}$. Then s forces that $\kappa \in \tilde{j}Y_n$, and that $\theta_{f_n} = \tilde{j}f_n(\kappa) < \tilde{j}f_{n+1}(\kappa) = \theta_{f_{n+1}}$. This contradiction completes the proof.

COROLLARY. Con(ZFC + *there exist two supercompact cardinals*) *implies* Con(ZFC + PFA + ω_1 *is precipitous*).

We give an example of a partial ordering $\mathcal{P}$ in $V[G_{\mathcal{L}^\kappa}]$ such that in $V[G_{\mathcal{L}^\kappa * \mathcal{P}}]$ the master condition ideal of Lemma 2 is not precipitous. Let $\mathcal{Q}$ be the usual partial ordering for adding a $g \in (\omega_1)^{\omega_1}$ which eventually dominates each ground model $h \in (\omega_1)^{\omega_1}$. For $A \subseteq \omega_2$ let $\mathcal{Q}_A$ be the side by side product, with countable supports, of A copies of $\mathcal{Q}$.

THEOREM. *In $V[G_{\mathcal{L}^\kappa * \mathcal{Q}_{\omega_2}}]$ let $\mathcal{I}$ be the ideal defined as in Lemma 2 with respect to a sufficiently closed elementary embedding $j: V \to M$ with critical point κ. Then $\mathcal{I}$ is not precipitous.*

PROOF. For $\beta < \kappa^+ = \omega_2^{V[G_{\mathcal{L}^\kappa}]}$, let g_β be the dominating function added by the βth copy of $\mathcal{Q}$. Let $\mathcal{R} = \mathcal{L}^\kappa * \mathcal{Q}_{\omega_2}$.

LEMMA 1. *For each $r \in \mathcal{R}$ and $\dot{X}$ such that $r \Vdash \dot{X} \in \mathcal{I}^+$, each $\beta < \kappa^+$, and each $\theta < j\kappa$, jr may be extended in $j\mathcal{R}$ to force that $j^{-1}{}''(G_{j\mathcal{R}})$ is $\mathcal{R}$-generic, $\kappa \in \tilde{j}\dot{X}$, and $\tilde{j}g_\beta(\kappa) \geqq \theta$.*

PROOF. Let $r, \dot{X}, \beta, \theta$ be a counterexample. For some $A \in [\kappa^+]^\kappa$, $\dot{X} \in V^{\mathcal{L}^\kappa * \mathcal{Q}_A}$. By the automorphism properties of $\mathcal{Q}_{\omega_2}$ we may assume that $\beta = 0$ and $A = \kappa$.

Denote $j \upharpoonright V_{\kappa+3}$ by $\hat{j}$. Then M satisfies

> $\hat{j}: V_{\kappa+3} \to V_{j\kappa+3}$ is an elementary embedding, and $\hat{j}r$, $[\![\hat{j}^{-1}{}''(G_{j\mathcal{R}})$ is $\mathcal{R}$-generic$]\!]$, and $[\![\kappa \in \hat{j}X]\!]$ are mutually compatible and their conjunction forces that $\hat{j}g_0(\kappa) < \theta$.

For α inaccessible $< \kappa$, let $\mathcal{R}_\alpha = \mathcal{L}^\alpha * \mathcal{Q}_{\alpha^+}$. Then for a j-measure one set of α's there exist $\theta_\alpha < \kappa$ and j_α so that

> $\hat{j}_\alpha: V_{\alpha+3} \to V_{\kappa+3}$ is an elementary embedding and r, $[\![\hat{j}_\alpha^{-1}{}''(G_{\mathcal{R}})$ is $\mathcal{R}_\alpha$-generic$]\!]$ and $[\![\alpha \in \dot{X}]\!]$ are mutually compatible and their conjunction forces that $g_0(\alpha) < \theta_\alpha$.

Let $j_\kappa = j\langle j_\alpha : \alpha < \kappa\rangle(\kappa)$ and $\theta_\kappa = j\langle \theta_\alpha : \alpha < \kappa\rangle(\kappa)$. Again, M satisfies

> $j_\kappa: V_{\kappa+3} \to V_{j\kappa+3}$ is an elementary embedding, and jr, $[\![j_\kappa^{-1}{}''(G_{j\mathcal{R}})$ is $\mathcal{R}$-generic$]\!]$ and $[\![\kappa \in j\dot{X}]\!]$ are mutually compatible and their conjunction forces that $\tilde{j}g_0(\kappa) < \theta_\kappa$.

CLAIM. *If G is $\mathcal{R}$-generic then $j''(G)$ may be extended to a $j\mathcal{R}$-generic $G_{j\mathcal{R}}$ such that $[\![\kappa \in j\dot{X}]\!] \in G_{j\mathcal{R}}$ and such that $j_\kappa^{-1}{}''(G_{j\mathcal{R}})$ is $\mathcal{R}$-generic.*

PROOF. First note that for each $f \in (\kappa)^\kappa \cap V$ there is an $h \in (\kappa)^\kappa \cap V$ such that $j_\kappa f(\gamma) < jh(\gamma)$ for all $\gamma < j(\bar{\kappa})$. Namely, for each $\alpha < \kappa$ choose $h(\alpha)$ greater than $f(\alpha)$ and greater than $[j_\beta(f \upharpoonright \beta)](\alpha)$ for each $\beta \leqq \alpha$. This fact yields that if I is $j(\mathcal{L}^\kappa * \mathcal{Q})$-generic and $j^{-1}{}''(I)$ is $(\mathcal{L}^\kappa * \mathcal{Q})$-generic then $j_\kappa^{-1}{}''(I)$ is $(\mathcal{L}^\kappa * \mathcal{Q})$-generic.

If the claim were false, an $r \in \mathcal{R}$ would force its falsity, where we write $r = (l, q_0, q_1) \in \mathcal{L}^\kappa * (\mathcal{Q}_\kappa \times \mathcal{Q}_{\kappa^+ - \kappa})$. Pick an H containing $j(l, q_0)$, generic over $j(\mathcal{L}^\kappa * \mathcal{Q}_\kappa)$, such that $j^{-1}{}''(H)$ is $(\mathcal{L}^\kappa * \mathcal{Q}_\kappa)$-generic and such that $[\![\kappa \in jX]\!] \in H$.

We have that $j^{-1}{}''(H) \in V[G_{j(\mathscr{L}^\kappa)}]$; living in $V[G_{j(\mathscr{L}^\kappa)}]$, extend $j^{-1}{}''(H)$ to a J which is $(\mathscr{L}^\kappa * \mathscr{Q}_{j''(\kappa^+) \cup j''_\kappa(\kappa^+)})$-generic. For each $(l, q) \in J$, replace each demand in q of the form "the $j(\alpha)$th dominating function dominates $f \in (\kappa)^\kappa$ beyond $\theta < \kappa$" by "the $j(\alpha)$th dominating function dominates jf beyond θ", to obtain a subset $\bar{J}$ of the poset $(j\mathscr{L}^\kappa * \mathscr{Q}_{j''(\kappa^+) \cup j''_\kappa(\kappa^+)})$. Assume J was chosen so that $q_1 \in j^{-1}{}''(\bar{J})$. We have that $j^{-1}{}''(H) \subseteq j^{-1}{}''(J)$, that $j^{-1}{}''(\bar{J})$ is $\mathscr{R}$-generic, and as in the preceding paragraph, that $j_\kappa^{-1}{}''$ (downwards closure of $\bar{J}$) is $\mathscr{R}$-generic. In $V[G_{j(\mathscr{L}^\kappa)}]$, let d be the conjunction of $\bar{J} \restriction \mathscr{Q}_{j(\kappa^+)}$. Then, since $j(\kappa) = j_\kappa(\kappa)$, $d \restriction \mathscr{Q}_{j(\kappa)}$ is the conjunction of $H \restriction \mathscr{Q}_{j(\kappa)}$. Thus $H \cup \{d\}$ may be extended to a $j\mathscr{R}$-generic set. This establishes the claim.

Pick $G_{j\mathscr{R}}$ as in the claim. Then in $M[G_{j\mathscr{R}}]$, $j^{-1}{}''(G_{j\mathscr{R}})$ is $\mathscr{R}$-generic and $\tilde{j}g_0(\kappa) < \theta_\kappa$. But for some $\alpha_0 < \kappa$ and all $\alpha > \alpha_0$ so that θ_α exists, $g_0(\alpha) > \theta_\alpha$. By elementarity, $\tilde{j}g_0(\kappa) > \theta_\kappa$. This contradiction establishes Lemma 1.

LEMMA 2. *Suppose $X \in \mathscr{I}^+$, $\sigma < \kappa^+$. Then for some $\beta < \kappa^+$,*

$$\{\alpha \in X : g_\beta(\alpha) < g_\sigma(\alpha)\} \in \mathscr{I}^+.$$

PROOF. Suppose $r \in \mathscr{R}$ forces that $\langle \dot{X}, \sigma \rangle$ is a counterexample. Pick $\beta > \sigma$ so that $\dot{X} \in V^{\mathscr{L}^\kappa * \mathscr{Q}_\beta}$. Let $G_{\mathscr{R}}$ be $\mathscr{R}$-generic with $r \in G_{\mathscr{R}}$. Let $\theta = \sup\{jf(\kappa) : f \in (\kappa)^\kappa \cap V\}$. Then $\theta < j(\kappa)$ and if H is any $j(\mathscr{L}^\kappa * \mathscr{Q}_\beta)$-generic set compatible with $j''(G_{\mathscr{R}})$, the condition $[\![g_{j\beta}(\kappa) = \theta]\!]$ is a nonzero member of $j\mathscr{R}/H \cup j''(G_{\mathscr{R}})$. Since $\dot{X} \in V^{\mathscr{L}^\kappa * \mathscr{Q}_\beta}$ there is, by Lemma 1, an H as above with $[\![\tilde{j}g_\sigma(\kappa) > \theta]\!] \in H$. Pick a $j\mathscr{R}$-generic $G_{j\mathscr{R}} \supseteq H \cup \{j''(G_{\mathscr{R}})\} \cup [\![\tilde{j}g_\beta(\kappa) = \theta]\!]$; then $G_{j\mathscr{R}}$ witnesses that $\{\alpha \in X : g_\beta(\alpha) < g_\sigma(\alpha)\} \in \mathscr{I}^+$.

Lemma 2 gives player II a strategy in the game

I	X_0		X_1		$\ldots$ $\begin{pmatrix} X_n \supseteq Y_n \supseteq X_{n+1} \in \mathscr{I}^+ \\ f_{n+1} <_{Y_{n+1}} f_n \end{pmatrix}$
II		(Y_0, f_0)		(Y_1, f_1)	

guaranteeing that the game does not terminate at some finite stage (player II picks each f_n from $\{g_\beta : \beta < \kappa^+\}$ so that each $Y_{n+1} = \{\alpha \in X_{n+1} : f_{n+1}(\alpha) < f_n(\alpha)\}$ is in $\mathscr{I}^+$). Thus $\mathscr{I}$ is not precipitous.

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